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Problem Set #7

(1)

Definition 2. A function f is said to be meromorphic in a domain D if at every point of D it is either analytic or has a pole

(a) $2z + z^3$ clearly this function is analytic in D consisting of all points of the complex plane thus it is *meromorphic*

(b) $Log\ z$ is analytic in the domain D consisting of all points of the complex plane except those lying on the non-positive real axis. Thus, it is *not meromorphic*

(c) $\frac{sinz}{z^3+1}$ is *meromorphic*, because sin(z) is analytic and $\frac{1}{z^3+1}$ has a pole at -1, thus definition 2 holds.

(d) $e^{1/z}$ has an essential singularity at 0, thus definition 2 does not hold, hence *not meromorphic*

(e) tan(z) = sin(z)/cos(z) where sin(z) is analytic and 1/cos(z) has a pole, thus Definition 2 holds hence *meromorphic*

(f) $\frac{2i}{(z-3)^2} + cosz$, this is analytic and has a pole of order 2 at z=3, thus definition 2 holds and hence *meromorphic*

(2)

We know that from Theorem 3 we can deduce that is f is analytic and nonzero at each point of a simple closed positively oriented contour C and is meromorphic inside C, then

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i[N_0(f) - N_p(f)]$$

In our case $f = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ thus, it has no poles, in other words, $N_p(f) = 0$, therefore

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i[N_0(f)]$$

By **example 4** in 6.7 we can conclude that $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ has n zeros (by applying theorem 4 and deducing that $|h(z)| < |f(z)|\ where\ f(z) = a_n z^n\ and\ h(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$)

In other words, $N_0(f) = n$, ths

$$\int_C \frac{f'(z)}{f(z)} dz = 2n\pi i$$

(4)

Here we can say that $g(z) = f(z) - w_0$ and $g'(z) = f'(z) + 0$ since w_0 *is just a constant*

Therefore by definition since f(z) is analytic on the closed disk and does not have any poles, we must deduce that:

$$\frac{1}{2\pi i}\int_C \frac{g'(z)}{g(z)} dz = [N_0(g)]$$

but g(z) is zero when $f(z) - w_0 = 0$, in other words whenever $f(z) = w_0$, therefore the integral above is equal to the number of solution of $f(z) = w_0$

(6)

$$g(z) = z^6 + 4z^2 - 1$$

We take C as the circle |z|=1, and we regard g as a perturbation of the function $f(z) = 4z^2$ which clearly has two zeroes inside (C). To test condition $|h(z)| < |f(z)|$ we estimate the perturbation $h(z) = z^6 - 1$ on C by $|h(z)| = |1 - 1| = 0$

which sure enough. is strictly less than $|f(z)| = |4z^2| = 4$, therefore g also has two zeroes inside |z|<1

(7)

Here we pick f(z)=27 which has zero roots. (i.e. it is a constant)

$$|h(z)| = |z^3 + 9z| \leq 8 + 18 = 26$$

$$since\ 26 < 27\ we\ conclude\ that\ g(z) has\ no\ roots\ in\ |z| < 2$$

(13)

The trivial demonstration is if we have $g(z) = z - z$ and we pick $f(z) = z$ and $h(z) = -z$ and let's say we are considering the disk |z|<2

Thus, $|f(z)| = 2\ and\ |h(z)| = 2\ and\ therefor\ |h(z)| \leq |f(z)|$ which is clearly not true, since the function g(z) has no root, it is simply 0!