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Problem Set #6

[Sect 4.3]

(1b)

$$\int_\Gamma e^z dz = \int_{-1}^{1} e^z dz = e^z\Big|_{-1}^{1} = e^1 - e^{-1} = 2\sinh(1)$$

(1e)

$$\int_\Gamma 1/z dz = \int_{-3i}^{3i} 1/z dz = Log(z)\Big|_{-3i}^{3i} = Log(|3|) + i\arg(3) - Log(|-3|) - i\arg(-3) = i\pi$$

(1g)

$$\int_\Gamma z^{1/2} dz = \int_{\pi}^{i} z^{1/2} dz = 2/3(z^{3/2})\Big|_{\pi}^{i} = 2/3(\pi^{3/2}) - 2/3(i^{3/2}) =$$

$$= 2/3(\pi^{3/2}) - (2/3(\sqrt{i})i) = 2/3(\pi^{3/2}) - (2/3(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)i) = 2/3(\pi^{3/2}) - \frac{\sqrt{2}}{3}i + \frac{\sqrt{2}}{2}$$

(2)

Claim: If P(z) is a polynomial and Γ is any closed contour, $\int_\Gamma f(z)dz = 0$

Explanation: Suppose that f(z) is a polynomial of degree 'n' : $f(z) = (z - z_0)^n$ then f(z) is the derivative of the function $F(z) = (z - z_0)^{n+1}/(n+1)$ which is analytic in the domain D consisting of all the points in the plane except $z = z_0$. So by Corollary 2 we can deduce that $\int_\Gamma f(z)dz = 0$ where Γ is a loop.

The intuition behind this is as follows: Since Γ is a loop we can break it up into two equal contours. In other words, the integral over these two equal contours will be the same. Thus, the integral over the whole loop must equal to 0!

(6)

As the problem states, first we compute the integral of the function $1/(z - z_0)$ along the portion of C from α to β.

$$\int_{\alpha}^{\beta} 1/(z-z_0) = Log(z-z_0)\Big|_{\alpha}^{\beta} = Log(\beta - z_0) - Log(\alpha - z_0) =$$

$$= Log(\frac{\beta - z_0}{\alpha - z_0})$$

Now as beta and alpha approach τ we have: (Noting that now we have to adjust the branch of Log since the principle branch of Log has a branch cut at 0.

$$Log(\frac{\beta - z_0}{\alpha - z_0}) = Log(\frac{\tau - z_0}{\tau - z_0}) = 2\pi i \qquad 0 < \arg z < 2\pi$$

(11)

We are asked to use the Theorem 6 on the function d(fg)/dz, so we will do just that:

$$\int_{zi}^{zt} d(fg)/dz = fg\Big|_{zi}^{dt}$$

However we also know that (fg)'=f'g+fg'. So, in order to get the integral of f'g we need to subtract the integral of fg'. In other words:

(fg)'-fg'=f'g, thus:

$$\int_{zi}^{zt} d(fg)/dz - \int_{\Gamma} f(z)g'(z)dz = f(z)g(z)\Big|_{zi}^{dt} - \int_{\Gamma} f(z)g'(z)dz$$

[Sec. 4.4]

(1a)

This contour is continuously deformable to Γ in D. This is evident because Γ_0 does not intersect the deleted points, and it is visually possible to imagine Γ_0 expanding to match the shape of Γ

(1c)

This contour also is continuously deformable to Γ. This is due to the fact that the circle has a huge radius, thus its range is past the deleted points. In other words we will not intersect them when deforming to Γ.

(10a)

Here, the domain of analytic everywhere except at $z = \pm 5$. The integral $\oint_{|z|=2} f(z)dz = 0$ is correct because $z = \pm 5$ lie exterior to the contour, thus the integral is 0 by Cauchy's integral theorem.

(10d)

In this case the domain is analytic when z > -3. The integral $\oint_{|z|=2} f(z)dz = 0$ is correct because z=3 lies exterior to the contour, thus the integral is 0 by Cauchy's integral theorem.

(11)

The function e^{z^2} is analytic. The whole plane C is simply connected. Thus by Theorem 10 we can say that e^{z^2} has an anti-derivative in the whole plane.

(14)

The easiest way to prove this, is if we divide C into three parts. Each part contains only C_1, C_2 or C_3. Having divided C into such parts, (contours) we can apply Theorem 13 and say that the integral of C consists of the sum of the three contours containing C_1, C_2 or C_3. Furthermore the integral of part that contains C_1 is equal to the integral of C_1 itself (By theorem 13). Likewise, the integral of the part containing C_2 is equal to the integral of C_2. Thus: $\int_C f(z)dz = \int_{C1} f(z)dz + \int_{C2} f(z)dz + \int_{C3} f(z)dz$

(16)

Looking at the function:

$$f(z) = \frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \ldots + \frac{A_1}{z^1} + g(z)$$

We can apply Theorem 13 and Theorem 9 and say that integral of g(z) over the circle |z|=1 is 0 (because g(z) is analytic inside and on the circle |z|=1. Same goes for $\frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}}$ (Not $\frac{A_1}{z^1}$), their integral over the circle is also 0 because they are analytic on the inside and on the circle |z|=1. However, the $\frac{A_1}{z^1}$ is not analytic at z=0 (inside of the circle) Thus, every partial fraction vanishes in f(z) except $\frac{A_1}{z^1}$. Thus using Theorem 13 we can say that the integral f(z) over the circle |z|=1 is equal to the integral of $\frac{A_1}{z^1}$ since every other partial fraction will equal to 0 (Theorem 9). In other words, integral f(z) over the circle |z| = 1 is $2\pi i A_1$

(17)

By computing partial fractions, we get:

$$\lim_{z->1} R(z)((z-1)^2) = 1 = A$$
$$\lim_{z->-1} R(z)((z+1)) = 1 = C$$
$$\lim_{z->1} \frac{\partial R}{\partial z} R(z)((z-1)^2) = 1 = B$$

Using Theorem 13 we can enclose the branch cut inside a circle. We will have two circles, (around two branch cuts), they will be going in opposite direction (Because two halves of the contour go in opposite direction). Since we traverse the contour once, the integral of our function over these two circles will have equal but opposite value. Thus, the total integral will equal 0!