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Problem Set #3

[1]

(a)

Recall that in lecture we have derived the following result:

$$\gamma(1) = E[AR(1)AR(t-1)] = E[AR(t-1)][b \times AR(t-1) + e(t)] = b \times \gamma(0) = b$$

Recall from above that AR(t-1) did not depend upon e(t) so that the covariance of AR(t-1) and e(t) is zero. Similarly, the autocovariance at lag two is:

$$\gamma(2) = E[AR(t)AR(t-2)] = E[AR(t-2)][b \times AR(t-1) + e(t)] = b \times \gamma(1) = b^2$$

and the autocorrelation function at lag two is b^2,

and in general:

$$\gamma(u) = b^u$$

Thus,

$\rho(0) = 1$

$\rho(1) = 0.5$

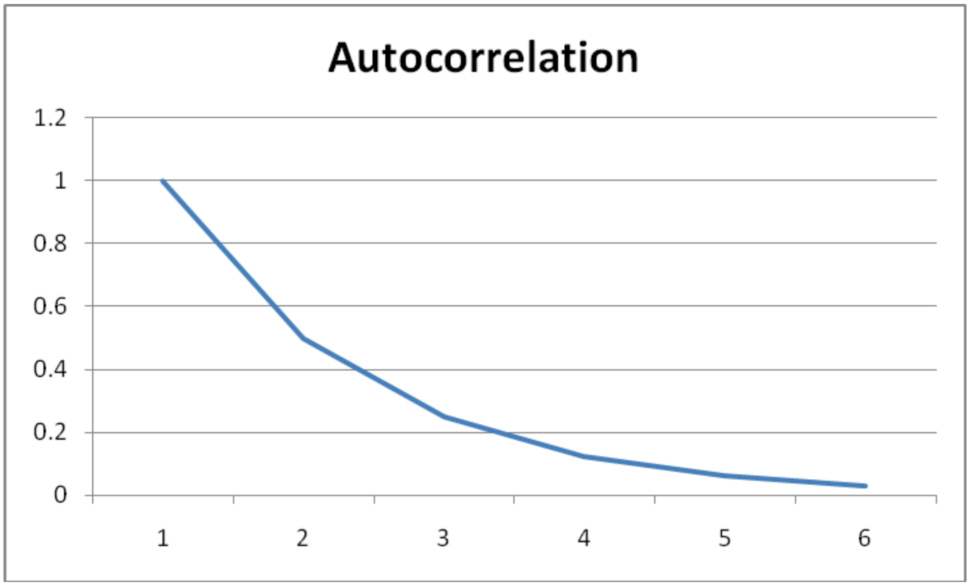
$\rho(2) = 0.25$

$\rho(3) = 0.125$

$\rho(4) = 0.0625$

$\rho(5) = 0.03125$

Plotting this we get:



(b)

We have the following formula:

x(t) = b_1x(t-1) + e(t), where b_1 in our case is 0.5

The partial autocorrelation function, $p(\tau)$, is just the coefficient of $y_{t-\tau}$ in a population linear regression of y_t on $y_{t-1}, \ldots.., y_{t-\tau}$

At lag(0) p(0) is simply x(t) (Since Zeroth derivative of x(t) is x(t))

Now, at lag(1),

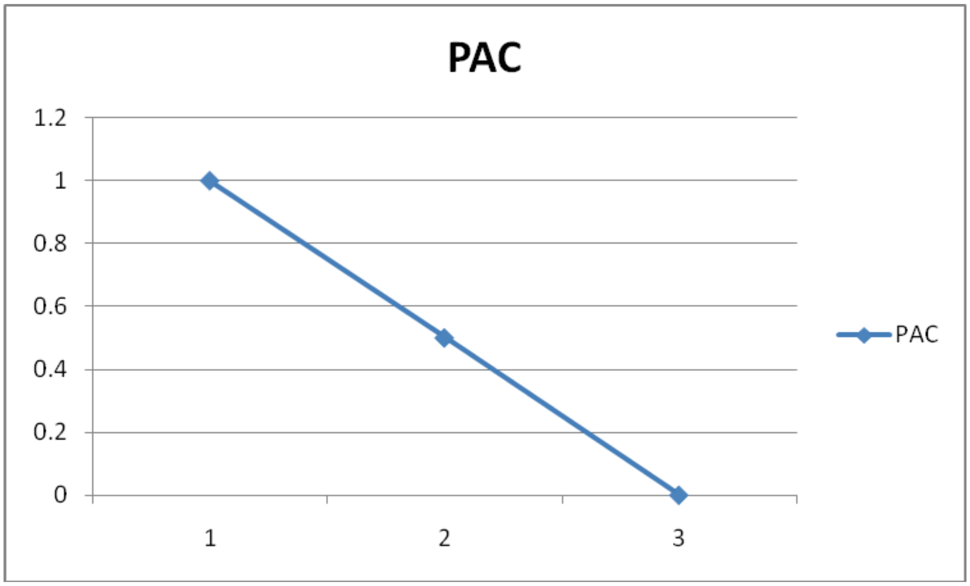
The change in the process for a change one period ago is:

$$\partial x(t)/\partial x(t-1) = b_1 = 0.5,$$

And from the Yule-Kendall equation for an AR(1) the estimate for b1 is:

$$b_1 = p(1),$$

Now applying the same strategy for partial auto-correlations we get 0. (i.e. the number of non-zero terms in our partial correlation function will actually determine the degree of AR!)



(2)

(a) Again, we have derived the autocorrelation function for an AR(2) in Problem Set #2.

From ideas above we can, for example, show that autocovariance at lag one is:

$$\gamma(t,1) = E\big(x(t)x(t-1)\big) = E[b_1x(t-1)^2 + b_2x(t-2)x(t-1) + \overbrace{e(t)x(t-1)}^{E.V.=0}]$$
$$= b_1\gamma(0) + b_2\gamma(1)$$

Similarly, we derive the autocvariance function at lag two:

$$\gamma(2) = E\big(x(t)x(t-2)\big) = E[b_1x(t-1)x(t-2) + b_2x(t-2)^2 + \overbrace{e(t)x(t-2)}^{E.V.=0}] = b_1\gamma(1) + b_2\gamma(0)$$

And finally, by recognizing recursion, we can derive autocovarience of lag three:

$$\gamma(3) = E\big(x(t)x(t-3)\big) = E[b_1x(t-1)x(t-3) + b_2x(t-2)x(t-3) + \overbrace{e(t)x(t-3)}^{E.V.=0}]$$
$$= b_1\gamma(2) + b_2\gamma(1)$$

Thus, clearly by multiplying by ARTWO(t-3) and taking expectations, we obtain a recursive formula for the autocovarience of lag 3 in terms of the autocovarience at lag 2 and the autocovarience at lag 1.

In order to derive $\rho(3)$ we simply divide by $\gamma(0)$ to obtain:

$$\frac{\gamma(3)}{\gamma(0)} = \frac{b_1\gamma(2) + b_2\gamma(1)}{\gamma(0)}$$

$$\rho(3) = b_1\rho(2) + b_2\rho(1)$$

We can further derive the Autocorrelation for AR(2) to be:

$$p_{xx(1)} = \frac{b_1}{(1 - b_2)}$$

$$p_{xx(U)} = b_1 p_{xx}(U - 1) + b_2 p_{xx}(U - 2)$$

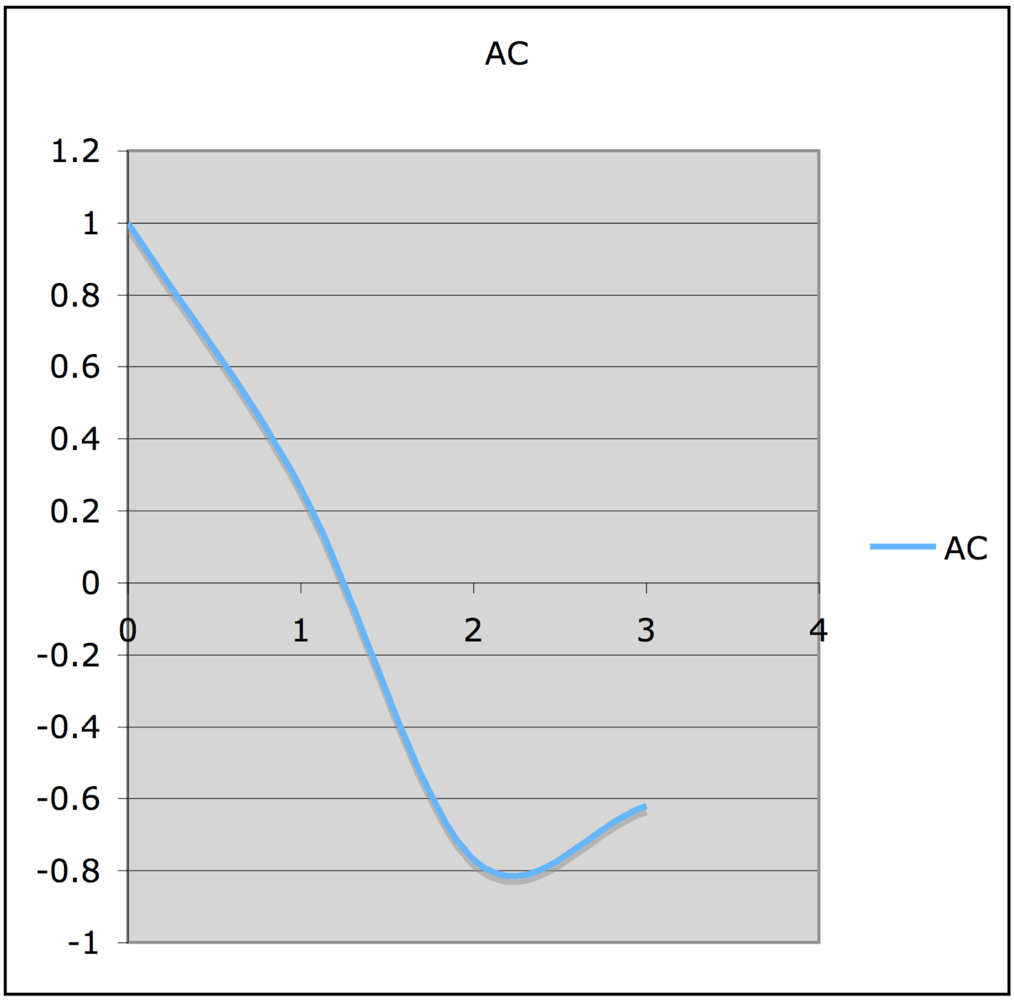
Thus,

$$\rho(0) = 1$$

$$\rho(1) = \frac{0.5}{(1 - (-0.9))} = 0.263$$

$$\rho(2) = 0.5p_{xx}(1) + (-0.9)p_{xx}(0) = -0.7684$$

$$\rho(3) = 0.5p_{xx}(2) + (-0.9)p_{xx}(1) = -0.6209$$



(b)

$x(t) = b_1 x(t-1) + b_2 x(t-2) + e(t)$,

the change in the process for a change two periods ago is:

$$\partial x(t)/\partial x(t-2) = b_2,$$

and from the Yule-Kendall equations for an AR(2) the estimate for b_2 is:

$$b_2 = \rho_{x,x}(2) - b_1[\rho_{x,x}(1)]^2,$$

and the estimate for b_1 is:

$$b_1 = \rho_{x,x}(1)(1-b_2),$$

and substituting for b_1 and solving for b_2, the estimate is:

$$b_2 = \{\rho_{x,x}(2) - [\rho_{x,x}(1)]^2\}/\{1- [\rho_{x,x}(1)]^2\} = PACF(2),$$

i.e. the estimate of the partial autocorrelation at lag two. Note that if the time series is an AR(1), then

$$\rho_{x,x}(2) = [\rho_{x,x}(1)]^2,$$

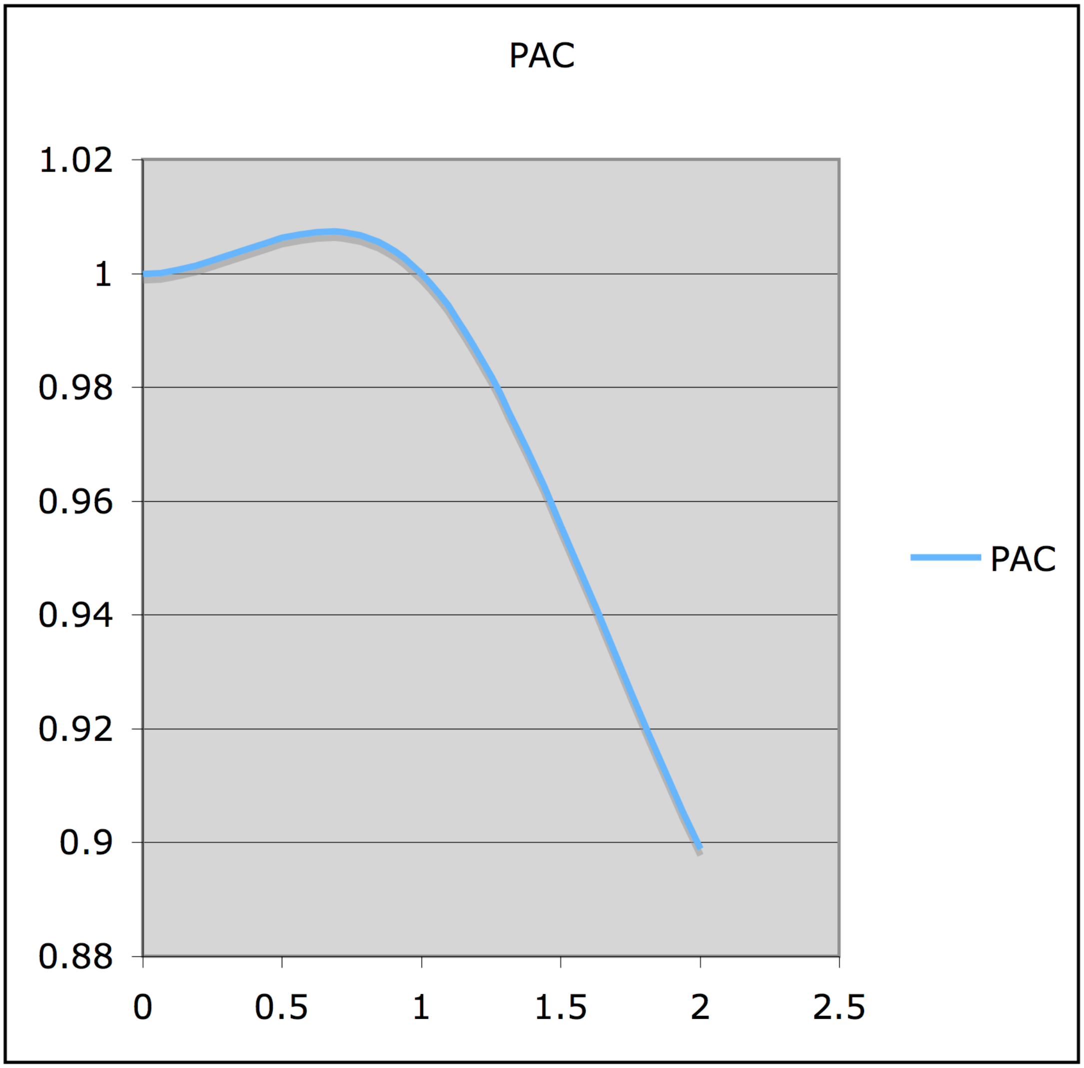
and the partial autocorrelation function at lag two will be zero, so that the number of non-zero estimates of the partial autocorrelation function will indicate the order of the autoregressive process.

PACS(0) =1

PACS(1) =1

$$PACS(2) = \frac{[ACF(2) - ACF(1)^2]}{[1 - ACF(1)^2]} = \frac{[-0.7684 - 0.263^2]}{[1 - 0.263^2]} =$$

= -0.899



(3) On the given plot we see that we have a slow decay, with an obvious spike at lag one in our partial autocorrelation. This would suggest that our process is AR(1).

Note*: Note that the decay starts after a few lags, which would indicate that there is a possibility of a mixed autoregressive model and moving average. In our example we are only provided with choice of AR(1) and AR(2) thus we pick AR(1).